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Topological Quantum Field Theories

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Chapter 1

Introduction

There are no landmarks in space; one portion of space is exactly like every other portion, so that we cannot tell where we are. We are, as it were, on an unruffled sea, without stars, compass, sounding, wind or tide, and we cannot tell in what direction we are going -Maxwell, Matter & Motion [M]

Classically there are two ways of describing the evolution of a mechanical system, the first proceeds from the equations of motion. A manifold is specified, the phase space of system, and given initial conditions, a trajectory, the worldline, is described on the manifold. This is the Hamiltonian approach, it singles out the time parameter. The second posits a functional called the action on all worldlines on the same manifold, and assumes that the actual worldline followed by the system is the one that minimises the action. This is the Lagrangian approach. It is manifestly covariant - all coordinates are treated democratically.

Both approaches have their counterparts in the quantum world. The first to be introduced was canonical quantization, the counterpart to the Hamiltonian approach. It proceeds by building a Hilbert space of possible quantum states of the system, and replacing dynamical variables by observables, which are operators on the Hilbert space, and whose expectation can be evaluated. In the Lagrangian approach, the basic object is the partition function of the theory which is expressed in terms of the Feynman path integral.

In a quantum field theory, the states are specified by fields on a background manifold, which is supplied with a metric. This background metric is basic to all constructions in the theory.

A Topological Quantum Field Theory (TQFT), is one in which the output of the theory is independent of the background metric. In particular, as the background metric is basic to all constructions in the quantum theory, what this means is that the output of the theory is not altered under any variations of the metric. It is said to have no local degrees of freedom. It an only depend on the the global shape of the manifold, this means it produces topological invariants of the manifold.

Following the discovery of a new knot invariant, the Jones polynomial, Edward Witten produced a physical argument that this was an invariant of a TQFT, by considering Chern-Simons theory on the 3-Sphere, with gauge group SU(2), and calculating the vacuum expectation value of a Wilson loop, that is the trace of a holonomy of a loop on the sphere in the fundamental representation of SU(2).

In order to formalise Wittens work, Atiyah [A] proposed aset of axioms for a TQFT, inspired by an earlier definition by Segal [S] of a Conformal Field Theory. This did not explicitly mention corbordisms or of category theory, but remained implicit.

Category theory was cofounded by MacLane & Eilenberg, and originated in algebraic topology. Roughly speaking, it is the study of structure, and structure preserving maps. It was used by Grothendieck to completely refound the subject of algebraic geometry and later seen to have a foundational impact on mathematics - it was noted that the theory provided an alternative to set theory as the basis for mathematics. In contrast to set theory, where the basic notion is that of set & membership; the basic notion of category theory is that of an object & maps betweens objects, with the emphasis on maps. Having originated in pure mathematics, it has had its deepest impact there; it was only realized much later that categories were implicit in many constructions in physics. Although the subject has a fearsomely abstract reputation, it relies on simple & intuitive ideas that are easy to motivate.

In this language, the axioms for a n-dimensional TQFT were seen to be equivalent to a functor, a *higher* level map, between the category of n-corbordisms to the category of finite-dimensional vector spaces.

Our goal is to understand what this means.

Chapter 2

Functorial TQFT

We begin by motivating Atiyahs axioms for a TQFT by a heuristic physical argument. We then state the axioms, and reformulate them categorically after introducing some of the language of category theory. We then draw out a few simple conclusions.

2.1 Locality and the Path Integral

Our starting point is the Feynman path integral. The standard mathematical apparatus for integration theory uses the notion of a measure on the space. This allows access to clean limiting and swapping of integration arguments, and is capable of great generalisation. However Cameron proved that there can be no such measure for the Feynman path integral. (Kac showed that the 'wick-rotated' integral does, the Weiner measure on the space of all continuous paths, and inspired by Feynman showed that the diffusion equation had a solution using this integral).

There have been numerous attempts to place the integral in an alternative rigorous framework, however none mirror the good properties of the classical theory. Atiyahs suggestion was to use certain natural properties of the integral as its defining property.

Consider the following:

$$Z(M) = \int_{FM} e^{-S\phi} D\phi \tag{2.1}$$

Where FM is a space of fields on a manifold M, and $D\phi$ is a measure on this space, and $S : FM \to \mathbb{C}$ is the action functional.

What are '*fields*'? They are a physical notion, and for our purposes, are objects that live on manifolds and can be defined '*locally*'. For example we could take $FM = C^{\infty}(M, \mathbb{R})$ or $FM = \{\text{principal } G\text{-bundles on } M\}$ for some fixed lie group G.

What is '*locality*'? Manifolds are glueable, to give a manifold is to give an arbitrary decomposition with gluing maps on overlaps. This is the content of their standard construction via atlases of open sets and transition maps. And they are restrictable, any open set of a manifold gives another manifold. It is plausible that the notion can be extended to fields, and we extend it in the following sense:

Suppose $M = M_1 \cup M_2$. Where both submanifolds may have boundary.

Restriction: Given a field on a manifold, we expect that the restriction of the field to any submanifold is also a field on the submanifold. We do not say for any open set, as we will only have submanifolds of the same dimension as the ambient manifold. So, given a manifold M, and a submanifold N of M, then $\forall \phi \in FM, \phi|_N \in FN$.

Gluing: We cannot take the arbitrary union of any two manifolds, as the result may not have manifold structure. So, instead we look for a manifold which can be decomposed into the union of two submanifolds, and given such a decomposition, suppose we have a field each on two submanifolds whose restrictions to their common intersection (which will be of a lower dimension, if they both have boundary, and they intersect there) ,which exists by the previous, are equal, we expect that there is a field on their union, the original manifold. That is, given manifolds M, M_1, M_2 such that $M = M_1 \cup M_2$,then $\forall(\phi_1, \phi_2) \in FM_1 \times FM_2$ such that $\phi_1|_{M_1 \cap M_2} = \phi_2|_{M_1 \cap M_2}$, $\exists \phi \in FM$ such that $(\phi_1, \phi_2) = (\phi|_{M_1}, \phi|_{M_2})$. We denote this glued field ϕ as $\phi_1 \cup \phi_2$.

We then note that fixing a field $\phi \in F(M_1 \cap M_2)$, in general we have:

$$FM = \{(\phi_1, \phi_2) \in FM_1 \times FM_2 : \phi_1|_{M_1 \cap M_2} = \phi_2|_{M_1 \cap M_2}\} \tag{2.2}$$

Two cases are of special interest, when the manifolds are disjoint, we have

$$FM = F(M_1 \sqcup M_2) = FM_1 \times FM_2 \tag{2.3}$$

When the manifolds intersect only and wholly on their boundaries, so $N = M_1 \cap M_2 = \partial M_1 = \partial M_2$, we have

$$FM = \{(\phi_1, \phi_2) \in FM_1 \times FM_2 : \partial\phi_1 = \partial\phi_2\} \tag{2.4}$$

The action functional is defined on the set of fields, when can we expect it to be additive?

Action Additivity: Given a manifold M, and some proper submanifold N i.e. of strictly lower dimension, then the submanifold will have it's own measure μ_N, and its volume $\mu_N(N)$ in this measure is not zero, but using the measure μ_M of the ambient manifold, the volume $\mu_M(N)$ is zero. (ie a plane in 3-space has zero volume). Now suppose M is the union of two submanifolds M_1 & M_2, then given $\phi \in FM$, we have the restrictions $\phi_i := \phi|_{M_i} \in FM_i$, and there is no reason to suppose the action is additive, but supposing the intersection $M_1 \cap M_2$ is proper in both M_1 & M_2, and so of zero volume there, it is plausible to expect the action functional should be 'additive'.

$$S\phi = S(\phi_1 \cup \phi_2) = S(\phi_1) + S(\phi_2) \tag{2.5}$$

This is similar to the additive axiom in measure theory - a measure is additive on sets when the sets are disjoint. We note in passing, that FM could be considered as a type of infinite dimensional manifold, so we're looking for a gluing laws and measure theory that works for these kinds of spaces. None of this is rigorous, and should be seen as the context for the following also non-rigorous calculation.

Let us first fix some notation.

We set F^*M to be the vector space of all complex valued functions on FM. If $\phi \in FM$, then we write $\partial\phi := \phi|_{\partial M}$. By convention we fix the set of fields for the empty manifold to be some single element, so that $F^*\varnothing = \mathbb{C}$. We also fix the value of the integral of the empty function as zero.

Now,

$$
\begin{aligned}
Z(M) &= \int_{\phi \in FM} e^{-S\phi} D\phi \\
&= \iint_{\substack{(\phi_1,\phi_2) \in FM_1 \times FM_2 \\ \partial\phi_1 = \partial\phi_2}} e^{-S(\phi_1 \cup \phi_2)} D\phi_1 D\phi_2 \\
&= \iint_{\substack{(\phi_1,\phi_2) \in FM_1 \times FM_2 \\ \partial\phi_1 = \partial\phi_2}} e^{-S\phi_1} e^{-S\phi_2} D\phi_1 D\phi_2 \\
&= \int_{\phi \in F(M_1 \cap M_2)} D\phi \left(\iint_{\substack{(\phi_1,\phi_2) \in FM_1 \times FM_2 \\ \partial\phi_1 = \partial\phi_2 = \phi}} e^{-S\phi_1} e^{-S\phi_2} D\phi_1 D\phi_2 \right) \\
&= \int_{\phi \in F(M_1 \cap M_2)} D\phi \left(\int_{\substack{\phi_1 \in FM_1 \\ \partial\phi_1 = \phi}} e^{-S\phi_1} D\phi_1 \right) \left(\int_{\substack{\phi_2 \in FM_2 \\ \partial\phi_2 = \phi}} e^{-S\phi_2} D\phi_2 \right) \\
&= \int_{F(M_1 \cap M_2)} D\phi (K_{\partial M_1}\phi)(K_{\partial M_2}\phi)
\end{aligned}
\tag{2.6}
$$

Where we have defined for an arbitray manifold X, a functional of the fields on its boundary, $K_{\partial X} : F\partial X \to \mathbb{C}$ by

$$
K_{\partial X}(\phi) := \int_{\substack{\psi \in FX \\ \partial\psi = \phi}} e^{-S\psi} D\psi \tag{2.7}
$$

so $K_{\partial X}$ is in $F^*\partial X$. Hence we've shown:

$$
Z(M) = \int_{F(M_1 \cap M_2)} (contribution\ from\ M_1)(contribution\ from\ M_2) \tag{2.8}
$$

Where the contributions are elements of $F^*(\partial M_i)$.

We have supposed so far that the manifold M is without boundary, we now suppose it does. we fix orientations on the connected components of the boundary, and designate by $\partial^- M$ the union of all the negatively orientated boundary components, which we call the incoming boundary;and designate by $\partial^+ M$ the union of all the positively orientated boundary components, which we call the outgoing boundary so that $\partial M = \partial^+ M \sqcup \partial^- M$.

Then M is an example of a corbordism between manifolds $\partial^- M$ and $\partial^+ M$. By definition two manifolds are corbordant if there exists another manifold whose boundary is the disjoint union of the two.

Then the corbordism M gives a mapping $T_{\partial M} : F^*(\partial^- M) \to F^*(\partial^+ M)$ by

$$(T_{\partial M} f)\phi^+ := \int_{\phi^- \in F(\partial^- M)} (f\phi^-) K^X(\phi^- \sqcup \phi^+) \tag{2.9}$$

where $f \in F^*(\partial^- M)$ and $\phi^+ \in F(\partial^+ M)$.

2.2 Properties of $T_{\partial M}$

We now establish some properties of $T_{\partial M}$.

1. $K_{\partial M}$ is a specialisation of $T_{\partial M}$, since taking $\partial^+ M = \partial M, \partial^- M = \varnothing$, then $F(\partial^- M) = \mathbb{C}$, and so $(T_{\partial M} 1)\phi = K_{\partial M}\phi$, where 1 is the constant map of $\partial^- M$ to 1.

2. It is a linear operator, as $\forall f, g \in F^*(\partial^- M)$ and $\forall \phi \in F(\partial^+ M)$.

$$
\begin{aligned}
T_{\partial M}(f+g)(\phi) &= \int_{\psi\in F(\partial^- M)} (f+g)\psi K_{\partial M}(\phi\sqcup\psi) \\
&= \int_{\psi\in F(\partial^- M)} (f\psi)K_{\partial M}(\phi\sqcup\psi) + \int_{\psi\in F(\partial^- M)} (g\psi)K_{\partial M}(\phi\sqcup\psi) \\
&= (T_{\partial M}f)\phi + (T_{\partial M}g)\phi
\end{aligned} \tag{2.10}
$$

3. We have $(T_{\partial M}f)(\phi_2) = \int_{F(\partial^- M)} f(\phi_1)\bar{K}_{\partial M}(\phi_1,\phi_2)$, where we define $\bar{K}_{\partial M}(\phi_1,\phi_2) := K_{\partial M}(\phi_1\sqcup\phi_2)$. So $T_{\partial M}$ is an integral operator with kernel $\bar{K}_{\partial M}$,

4. Suppose X, Y, Z are closed manifolds of the same dimension, and $M : X \to Y, N : Y \to Z$ are corbordisms, then a corbordism $M+N : X \to Z$ can be obtained by gluing M and N together by identifying points of Y, the orientations on its boundary are those induced by those M and N, that is $\partial^-(M+N) = \partial^- M$, and $\partial^+(M+N) = \partial^+ M$, then $T_{\partial(M+N)} = T_{\partial M}\circ T_{\partial N}$ since first,

$$
\begin{aligned}
&K_{\partial(M+N)}(\phi^-,\phi^+) \\
&= \int\limits_{\substack{\phi\in F(M+N)\\ \partial\phi=(\phi^-,\phi^+)}} e^{-S\phi}D\phi \\
&= \int\limits_{\psi\in F(M\cap N)} D\psi \iint\limits_{\substack{\phi_M\in FM,\phi_N\in FN\\ \partial\phi_M=(\phi^-,\psi)\\ \partial\phi_N=(\psi,\phi^+)}} e^{-S(\phi_M\sqcup\phi_N)}D\phi_M D\phi_N \\
&= \int\limits_{\psi\in F(M\cap N)} D\psi \int\limits_{\substack{\phi_M\in FM,\\ \partial\phi_M=(\phi^-,\psi)}} e^{-S\phi_M}D\phi_M \int\limits_{\substack{\phi_N\in FN\\ \partial\phi_M=(\psi,\phi^+)}} e^{-S\phi_N}D\phi_N \\
&= \int\limits_{\psi\in F(M\cap N)} D\psi\ K_{\partial M}(\phi^-\cup\psi)K_{\partial N}(\psi\cup\phi^+)
\end{aligned} \tag{2.11}
$$

and so,

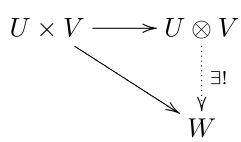
$$
\begin{aligned}
&(T_{\partial(M\sqcup N)}f)\phi^+ \\
&= \int\limits_{\phi^-\in F\partial^- M} D\phi^- f\phi^- K_{\partial(M\cup N)}(\phi^- \cup \phi^+) \\
&= \int\limits_{\phi^-\in F\partial M} D\phi^-\ f\phi^- \int\limits_{\psi\in F\partial^- N} D\psi\ K_{\partial M}(\phi^- \sqcup \psi) K_{\partial N}(\psi \sqcup \phi^+) \\
&= \int\limits_{\psi\in F\partial^+ M} D\psi \int\limits_{\phi^-\in F\partial^- N} D\phi^-\ f\phi^- K_{\partial M}(\phi^- \sqcup \psi) K_{\partial N}(\psi \sqcup \phi^+) \\
&= \int\limits_{\phi^-\in F\partial^- N} D\phi^-\ (T_{\partial M} f)\psi K_{\partial N}(\psi \sqcup \phi^+) \\
&= [T_{\partial N}(T_{\partial M} f)]\phi^+
\end{aligned}
\tag{2.12}
$$

5. Given disjoint corbordisms M and N, we claim $T_{\partial(M\sqcup N)} = T_{\partial M} \otimes T_{\partial N}$.

   First, the kernel is easily seen to be multiplicative on disjoint union, for suppose we have fields $(\phi, \psi) \in F\partial M \times F\partial N$, then

$$
\begin{aligned}
K_{\partial(M\sqcup N)}(\phi \sqcup \psi) &= \int_{\substack{\phi'\sqcup\psi'\in F(M\sqcup N)\\ \partial(\phi'\sqcup\psi')=\phi\sqcup\psi}} e^{-S(\phi'\sqcup\psi')} D(\phi' \sqcup \psi') \\
&= \int_{\substack{\phi'\in FM\\ \partial\phi'=\phi}} e^{-S\phi'} D\phi' \int_{\substack{\psi'\in FN\\ \partial\psi'=\psi}} e^{-S\psi'} D\psi' \\
&= K_{\partial M}(\phi) K_{\partial N}(\psi)
\end{aligned}
\tag{2.13}
$$

   We also show $F^*(M \sqcup N) \approxeq F^*M \otimes F^*N$. We establish this by using the universal property of the tensor product:



(2.14)

   That is any bilinear map between UV and W establishes an isomorphism between $U \otimes V$ and W, also if $U' \times V' \to U' \otimes V'$ is another tensor product, then the following commutes

$$\begin{array}{ccc} U \times V & \longrightarrow & U \otimes V \\ \Big\downarrow{\scriptstyle f \times g} & & \Big\downarrow{\scriptstyle f \otimes g} \\ U' \times V' & \longrightarrow & U' \times V' \end{array} \tag{2.15}$$

maps f and g determine a unique map $f \otimes g$.

clearly the map $\Phi_{M,N} : F^*M \times F^*N \to F^*(M \sqcup N)$ defined by

$$\Phi(f,g)(\phi) := f(\phi|_M)g(\phi|_N) \tag{2.16}$$

is bilinear, and we've already established that $F(M \sqcup N) = FM \times FN$, so the result follows.

Set $\Phi := \Phi_{\partial^- M, \partial^- N}$. Choose $\phi \in F\partial^+(M \sqcup N)$, and set $(\phi', \phi'') := (\phi|_{\partial^+ M}, \phi|_{\partial^+ N})$. Also choose $(f,g) \in F\partial^+ M \times F\partial^+ M)$.

$$\begin{aligned}
&(T_{M \sqcup N})[\Phi(f,g)]\phi \\
&= \int_{\psi \in F\partial^-(M \sqcup N)} \Phi(f,g)\psi K_{\partial(M \sqcup N)}(\psi \sqcup \phi) D\psi \\
&= \iint_{\substack{\psi' \in F\partial^- M \\ \psi'' \in F\partial^- N}} f(\psi')g(\psi'')K_{\partial M}(\psi' \sqcup \phi')K_{\partial N}(\psi'' \sqcup \phi'')D\psi' D\psi'' \\
&= \int_{\psi' \in F\partial^- M} f(\psi')K_{\partial M}(\psi' \sqcup \phi')D\psi' \int_{\psi'' \in F\partial^- N} g(\psi'')K_{\partial N}(\psi'' \sqcup \phi'')D\psi'' \\
&= (T_{\partial M} f)(\phi)(T_{\partial N} g)(\phi'') \\
&= \Phi(T_{\partial M} f, T_{\partial N} g)\phi
\end{aligned} \tag{2.17}$$

and consider the following diagram:

$$\begin{array}{ccc} F^*\partial^- M \times F^*\partial^- M & \xrightarrow{\Phi} & F^*\partial^-(M \sqcup N) \\ \downarrow{\scriptstyle T_{\partial^- M} \times T_{\partial^- N}} & & \downarrow{\scriptstyle T_{\partial(M \sqcup N)}} \\ F^*\partial^+ M \times F^*\partial^+ M & \xrightarrow{\Phi} & F^*\partial^+(M \sqcup N) \end{array} \tag{2.18}$$

by the above it commutes, and hence by the universal property of the tensor product we get $T_{\partial M} \otimes T_{\partial N} \cong T_{\partial(M \sqcup N)}$.

We summarise our results:

1. $T_{\partial M}$ is a linear map
2. $T_{\partial(M+N)} = T_{\partial M} \circ T_{\partial N}$ (Functoriality)
3. $T_{\partial(M \sqcup N)} = T_{\partial M} \otimes T_{\partial N}$ (Multiplicative)

2.3 Atiyahs Axioms

In the 1988 paper [A], Atiyah introduces his axioms for a finite dimensional TQFT. As his work is aimed at mathematicians, he works over a general ring. As our work is physically motivated we work over either the real or complex field.

A n-dimensional TQFT consists of the following data:

1. A finite dimensional vector space $Z(A)$ associated to each oriented closed smooth n-dimensional manifold A.

2. An element $Z(A) \in Z(\partial M)$ associated to each oriented smooth $(n+1)$-dimensional manifold (with boundary) M.

They are subject to the following laws:

(a) Z is functorial with respect to orientation preserving diffeomorphisms of A and A'.

(b) Z is involutory, i.e. $Z^* = Z(A)*$ where $A*$ is A with opposite orientation and $Z(A)*$ denotes the dual vector space.

(c) Z is multiplicative, this means for disjoint A_1 and A_2, we have $Z(A_1 \sqcup A_2) = Z(A_1) \otimes Z(A_2)$.

Functoriality means that an orientation preserving diffeomorphism $f : A \rightarrow A'$ induces a linear isomorphism $Zf : ZA \rightarrow ZA'$, and given $g : A' \rightarrow A''$, we also have $Z(gf) = ZgZf$.

Now a category, speaking roughly, is a collection of objects together with maps that preserve the structure of the objects. For example manifolds, together with smooth maps between them; or topological spaces together with homeomorphisms. Maps may not always compose, but when they do, they compose associatively. Categories themselves can be seen as the objects of some 'meta-category', and the maps between them should preserve the composition law, they're called functors. In this language, Vec is the category of all finite-dimensional spaces, with linear maps between them. And $nCob$ is the category of all n-dimensional corbordisms with smooth maps between them.

Then Atiyahs axioms can be stated simply and precisely as asserting the existance of a functor $Z : nCob \rightarrow Vec$.

Chapter 3

Category Theory

3.1 Axioms

We now formally introduce category theory:

Definition

1. A category A has a classes of objects $Obj\,A$, and a class of morphisms $Mor\,A$
2. There are two mappings $dom, cod : Mor\ A \to Obj\ A$, called the domain & codomain mapping respectively.
3. There is one mapping $id : Obj\ A \to Mor\ A$, called the identity map
4. There is a partially defined map $\circ : Mor\ A \times Mor\ A \to Mor\ A$, called composition, and generally written infix

Before we set out the axioms that these structures need to satisfy, we introduce some terminology. For a morphism (we also call them arrows) $f \in Mor\,A$, let $a := dom\ f$ and $b := cod\ f$, and we write $f : a \to b$. We call a the domain of f, and b the codomain of f. For an object $a, b \in Obj\ A$, we define $hom(a, b) := \{f \in Mor\ A : dom(f) = a \text{ and } cod(f) = b\}$, and we often write $id_a := id(a)$. The axioms are:

1. (transitivity) $\circ : hom(a,b) \times hom(b,c) \to (a,c)$, for any objects a, b, c

2. (associativity) $(f \circ g) \circ h = f \circ (g \circ h)$ when the composition is defined.

3. (left & right identity) for $f : a \to b \in Mor\ A$, we have $id_b \circ f = f$ and $f \circ id_a = f$

We make a couple of simple observations: for an object a, we must have $id_a : a \to a$ to satisfy the identity laws, and also indentities are unique by a variation of a standard argument: Suppose $\exists id'_a \in Mor\, A$, such that $f \circ id'_a = f$ for all composable $f \in Mor : A$, then $id_a \circ id'_a = id_a = id'_a$, and similarly for the right identity.

We think of composition in a category as pasting together 1-dimensional arrows, where arrows can be pasted together so long as they have compatible endpoints, associativity just means we're not concerned with the order in which the arrows are pasted together. With this language, we can now consider 2-dimensional arrows, that is an arrow between 1-dimensional arrows. The appropriate context for this discussion is a (strict) 2-category, but before we motivate their construction, we need to introduce functors as the structure-preserving maps between categories.

note:We will write $a \in A$ for $a \in Obj\ A$, and if the context is clear, we may do the same for morphisms. Generally objects will be labelled by lower-case letters from the beginning of the alphabet, and morphisms by lower-case letters from the middle).

3.2 Functors

What structure should a structure preserving map between categories preserve? Essentially we need to preserve composition & the identity. It turns out the appropriate definition is the following.

Given arbitrary categories A, B, then a functor $F : A \to B$, maps objects of A to objects of B, and arrows of A to arrows of B, such that

$$F : a \xrightarrow{f} b \xrightarrow{g} c \Rightarrow Fa \xrightarrow{Ff} Fb \xrightarrow{Fg} Fc \tag{3.1}$$

$$F : 1_a \Rightarrow 1_Fa \tag{3.2}$$

where $a, b, c \in A$.

There is a natural notion of composition of functors. For functors $F : A \to B, G : B \to C$ where $A, B, C \in Cat$, define $GF : A \to C$ by

$$\begin{aligned} (GF)a &:= G(Fa) \ \ \text{(on objects)} \\ (GF)f &:= G(Ff) \ \ \text{(on morphisms)} \end{aligned} \tag{3.3}$$

GF is a functor, since $(GF)(gf) = G(Fgf) = G(Fg \circ Ff) = G(Fg) \circ G(Ff) = (GF)f \circ (GF)f$, and $GF1_a = G(F1_a) = G1_{Fa} = 1_{(GF)a}$.

Composition is easily seen to be associative: Given $F : A \to B, G : B \to C, H : C \to D$ where $A, B, C, D \in Cat$ Then $((HG)F)a = (HG)(Fa) = H(G(Fa)) = H((GF)a) = (H(GF))a$, and similarly for morphisms. We designate all functors between A and B as $Fun[A, B]$, or simply as B^A.

We define Cat as the set of all categories. To now give Cat itself the structure of a category, we need to prescribe the morphisms from, say from category A to B, we take these to be all functors between them, that is B^A, and also we need to prescribe how they compose - we use the definition set out above, which we have shown obeys the composition law required for a category. We now only need to show the identity morphisms exist. This is easy, for any $A \in Cat$, define the identity functor $1_A : A \to A$ by

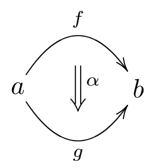
$$\begin{aligned}(1_A)a &:= a\text{on objects} \\ (1_A)f &:= f\text{on morphisms}\end{aligned} \tag{3.4}$$

Showing that this defines a functor is trivial: $1_A(gf) = gf = (1_A g)(1_A f)$ for $g, f \in A$, and $1_A 1_a = 1_a = 1_{1_A a}$. We claim that these functors are the identity morphisms in Cat:

Given a functor $F : A \to B$ with $A, B \in Cat$, then for all $a \in A$, we have $(1_B F)a = 1_B(Fa) = Fa \Rightarrow 1_B F = F$ on objects, and for all morphisms f in A, $(1_B F)f = 1_B(Ff) = Ff \Rightarrow 1_B F = F$ on morphisms, hence $1_B F = F$ identically. A similar argument establishes that $F1_A = F$. Hence the identity axiom is satisfied.

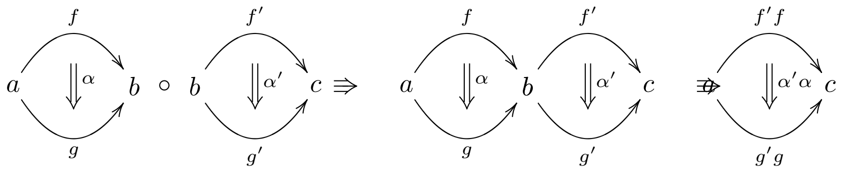
3.3 Strict 2-Categories

Suppose we have two arrows f, g between the same two objects,say a, b in a category, we suppose we have a '2-arrow' $\alpha : f \to g$. We illustrate this in

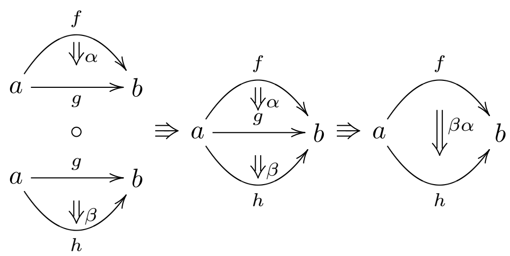


What would be the appropriate generalisation of the category to this context? Essentially we need to think about composition and identities. We take the previous illustration seriously, and think of pasting those '2-cells' in a 2-dimensional context. That is we can paste along one of the boundaries, this gives us a vertical composition, or at one of the end-points, which gives a horizontal composition. That is:

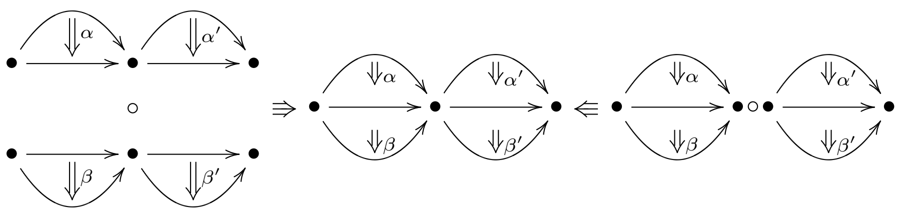
given $alpha' : f' \to g'$, where $f', g' : b \to c$, we have the following diagram



given $\beta : g \to h$, where $h : a \to b$, we have the following diagram

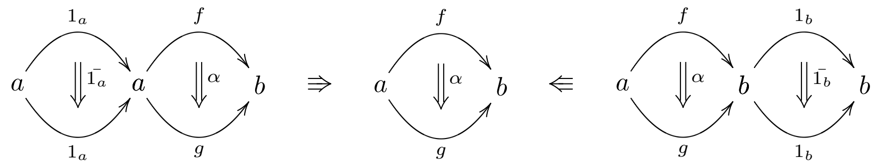


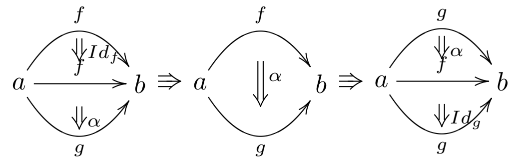
A natural generalisation of associativity for a 2-category, is that the order of pastings in a diagram should be immaterial. This is equivalent to stating that the order of horizontal pastings, and that of vertical pastings is immaterial plus they 'commute' - that is we have what is known as the *interchange* law.



It is natural to require that the 2-cells have identities for both compositions. This turns 2-cells under horizontal and vertical composition separately into a category.

By the uniqueness property of identities for 1-arrows, the horizontal identity 2-cell must have the appropriate identity morphism on both boundaries.





We note that the maps *dom* and *cod* are now functors for horizontal composition as they obviously preserve the boundary composition. We also have a new mapping Id which maps from 1-morphisms to 2-morphisms, we demand that this is also a functor, which essentially means that the horizontal composite of two vertical identities is itself a vertical identity, and we can then construct a composition bifunctor. These considerations motivates the formal definition of a strict 2-category A [Mac]:

Definition

1. a set of objects $a, b, c \cdots$
2. a function which assigns to each ordered pair of objects (a, b) a category $A(a, b)$
3. for each ordered triple (a, b, c), a bifunctor called composition $K_{a,b,c} : A(b, c) \times A(a, b) \to A(a, c)$
4. for each object a, a functor $Id_a : 1 \to A(a, a)$, (where 1 is the trivial category with 1 object and its identity).

Its possible to rephrase the definition of a category in terms of *hom*, which brings out the similarity of the definition of a strict 2-category to that of a category [Mac] I.8. In fact, the similarities lie much deeper, an ordinary category has *hom* valued in the category of Set, it is possible to rephrase the definition of a category, so that it can be valued in some other category, and this leads to the theory of enriched category theory [Kel]. In this language, a strict 2-category is precisely a category enriched in Cat, the category of all categories (we're ignoring foundational issues attached to the use of *all*), and a recursive definition of higher strict n-categories is made: a strict (n+1)-category is a n-category enriched in Cat.

3.4 Natural Transformations

Having introduced functors, a natural question to ask is whether $B^A := Fun[A, B]$ organises itself into a category. In fact it does with the appropriate notion of morphism between functors. They are called natural transformations and are a pervasive concept in category theory. In fact, Mac Lane [Mac] observes that ”’category’ had been defined in order to define ’functor’, and ’functor’ had been defined in order to be able to define ’natural transformation’”’.

Definition:

Let $F, G \in B^A$, then $\alpha : Obj\ A \to Mor\ B$ is a natural transformation between F & G when $\forall f : a \to b \in Mor\ A$ the following diagram commutes

$$\begin{array}{ccc} Fa & \xrightarrow{Ff} & Fb \\ {\scriptstyle \alpha a}\downarrow & & \downarrow{\scriptstyle \alpha b} \\ Ga & \xrightarrow[Gf]{} & Gb \end{array} \tag{3.5}$$

We write $\alpha : F \to G$.

3.4.1 Composition by functors

We observe that pre & post-composition of a natural transformation by a functor results in another natural transformation. Let $\alpha : F \to G$ where $F, G : A \to B$, for any categories $A, B \in Cat$. Choose some other category C, and some functor $H : B \to C$, and define $H\alpha : HF \to HG$ by

$$\forall a \in A, (H\alpha)a := H(\alpha a) \tag{3.6}$$

We show that it is a natural transformation, that is we show the following diagram commutes

$$\begin{array}{ccc} HFa & \xrightarrow{HFf} & HFb\bar{a} \\ {\scriptstyle H\alpha' a}\downarrow & & \downarrow{\scriptstyle H\alpha' b} \\ HGa & \xrightarrow[HGf]{} & HGb \end{array} \tag{3.7}$$

well,

$$\begin{aligned} (HGf)(H\alpha a) &= H(Gf \circ \alpha a) \\ &= H(\alpha b \circ Ff) \\ &= H\alpha b \circ HFf \end{aligned} \tag{3.8}$$

a similar observation shows that any functor $H : C \to A$ for some other category C defines a natural transformation $\alpha H : FH \to GH$ by

$$\forall a \in A, (\alpha H)a := (\alpha H a) \tag{3.9}$$

3.4.2 Vertical composition of natural transformations

Given natural transformations $\alpha : F \to G, \beta : G \to H$ for functors $F, G, H \in B^A$, composition $\beta\alpha : F \to H$ is

$$(\beta\alpha)(a) = (\beta a)(\alpha a) \quad \forall a \in A \tag{3.10}$$

and since the following diagram commutes, then this construction does in fact result in a well-defined natural transformation.

$$\begin{array}{ccc} Fa & \xrightarrow{Ff} & Fb \\ {\scriptstyle \alpha a}\downarrow & & \downarrow{\scriptstyle \alpha b} \\ Ga & \xrightarrow[Gf]{} & Gb \\ {\scriptstyle \beta a}\downarrow & & \downarrow{\scriptstyle \beta b} \\ Ha & \xrightarrow[Hf]{} & Hb \end{array} \tag{3.11}$$

3.4.3 The functor category

To identify B^A as a category, we need to show the associativity of morphisms, and the existance of identity morphisms.

Now given $\gamma, \beta, \alpha \in B^A$ we have

$$\begin{aligned} ((\gamma\beta)\alpha)(a) &= ((\gamma a)(\beta a))(\alpha a) \\ &= (\gamma a)((\beta a)(\alpha a)) \\ &= (\gamma)(\beta\alpha)(a) \end{aligned} \tag{3.12}$$

using the associativity of morphisms in the B, hence natural transformations are associative.

For any functor $F \in B^A$ we define a natural transformation $1_F : F \to F$ by

$$1_F a := 1_{Fa} \tag{3.13}$$

This is a well-defined natural transformation since the following obviously commutes

$$\begin{array}{ccc} Fa & \xrightarrow{Ff} & Fb \\ {\scriptstyle 1_F a = 1_{Fa}} \downarrow & & \downarrow {\scriptstyle 1_F b = 1_{Fb}} \\ Fa & \xrightarrow[Ff]{} & Fb \end{array} \tag{3.14}$$

and we show that these natural transformations are identity morphisms in B^A: choose some $\alpha : F \to G \in Mor\ B^A$, then for all $a \in A$, we have $\alpha a : Fa \to Ga$, and then

$$\begin{aligned} (\alpha 1_F)(a) &= (\alpha a)(1_F a) \\ &= (\alpha a)(1_F a) \\ &= \alpha a \end{aligned} \tag{3.15}$$

an analagous argument works on the left.

3.4.4 Horizontal composition of natural transformations

There is in fact another notion of composition for natural transformations, which we shall call horizontal composition (which we denote by *), the previous naturally will be called vertical composition (which will be denoted by juxtaposition).

For $F, G \in Fun[A, B]$ and $F', G' \in Fun[A', B']$ where the categories are arbitray, suppose we have natural transformations $\alpha : F \to G, \alpha' : F' \to G'$, then define

$$(\alpha' * \alpha)a := (\alpha'_G a)(F'\alpha a) \tag{3.16}$$

The maps are well-defined, as $\alpha a : Fa \to Ga \Rightarrow (F'\alpha)a : (F'F)a \to (F'G)a$ and $(\alpha' G)a : (F'G)a \to (G'G)a$ since Ga is in B. Hence, $(\alpha'_G a)(F'\alpha a) : F'Fa \to F'Ga \to G'Ga$, and so $\alpha'\alpha$ does indeed map from $F'F \to G'G$.

We now show that it is a well-defined natural transformation. We have by definition, the following two commuting diagrams, for morphism $f : a \to \bar{a} \in A, g : b \to \bar{b} \in B$

$$\begin{array}{ccc} Fa & \xrightarrow{Ff} & F\bar{a} \\ {\scriptstyle \alpha a}\downarrow & & \downarrow{\scriptstyle \alpha\bar{a}} \\ Ga & \xrightarrow[Gf]{} & G\bar{a} \end{array} \tag{3.17}$$

$$\begin{array}{ccc} F'b & \xrightarrow{F'g} & F'\bar{b} \\ {\scriptstyle \alpha' b}\downarrow & & \downarrow{\scriptstyle \alpha'\bar{b}} \\ G'a & \xrightarrow[G'g]{} & G'\bar{b} \end{array} \tag{3.18}$$

We note that $Gf : Ga \to G\bar{a} \in B$, and substituting this in the second of the two commutative diagrams gives:

$$\begin{array}{ccc} F'Ga & \xrightarrow{F'Gf} & F'G\bar{a} \\ {\scriptstyle \alpha' Ga}\downarrow & & \downarrow{\scriptstyle \alpha' G\bar{a}} \\ G'Ga & \xrightarrow[G'Gf]{} & G'G\bar{b} \end{array} \tag{3.19}$$

We need to show

$$\begin{array}{ccc} F'Fa & \xrightarrow{F'Ff} & F'F\bar{a} \\ {\scriptstyle \alpha' \alpha a}\downarrow & & \downarrow{\scriptstyle \alpha' \alpha\bar{a}} \\ G'Ga & \xrightarrow[G'Gf]{} & G'G\bar{b} \end{array} \tag{3.20}$$

We now evaluate

$$\begin{aligned} (\alpha'\alpha\bar{a})(F'Ff) &= \alpha' G\bar{a} \circ F'\alpha\bar{a} \circ F'Ff \\ &= \alpha' G\bar{a} \circ F'(\alpha\bar{a} \circ Ff) \\ &= \alpha' G\bar{a} \circ F'(Gf \circ \alpha a) \\ &= \alpha' G\bar{a} \circ F'Gf \circ F'\alpha a \\ &= G'Gf \circ \alpha' Ga \circ F'\alpha a \\ &= G'Gf \circ \alpha' * \alpha a \end{aligned} \tag{3.21}$$

and so $\alpha' * \alpha$ is a well-defined natural transformation. In fact horizontal composition is associative:

For $F, G \in Fun[A, B]$, $F', G' \in Fun[A', B']$ and $F'', G'' \in Fun[A'', B'']$ where the categories are arbitray, suppose we have natural transformations $\alpha : F \to G, \alpha' : F' \to G', \alpha' : F' \to G'$, then we have $\alpha'\alpha : F'F \to G'G$, and $\alpha''\alpha' : F''F' \to G''G'$. and

$$
\begin{aligned}
\alpha''(\alpha'\alpha)a &= \alpha''G'Ga \circ F''\alpha'\alpha a \\
&= \alpha''G'Ga \circ F''(\alpha'Ga \circ F'\alpha a) \\
&= \alpha''G'Ga \circ F''\alpha'Ga \circ F''F'\alpha a
\end{aligned} \tag{3.22}
$$

but

$$
\begin{aligned}
(\alpha''\alpha')\alpha)a &= (\alpha''\alpha')Ga \circ F''F'\alpha a \\
&= \alpha''G'(Ga) \circ F''\alpha'(Ga) \circ F''F'\alpha a
\end{aligned} \tag{3.23}
$$

which matches.

We now show that natural transformations under the horizontal composition also has identities. That is given a natural Transformation $\alpha : F \to G$, we must show the existance of natural transformations 1_H for any functor H, such that $\alpha * 1_F = \alpha$ and $1_G * \alpha = \alpha$. We've already established shown that: 1_F are the identity morphisms on the category B^A, for any $F \in B^A$ (ie using vertical composition), and that 1_A is the identity morphism for Cat. Consider the natural transformation

$$
\bar{1}_A := 1_{(1_A)} : 1_A \to 1_A \tag{3.24}
$$

Recall horizontal composition is defined by $(\alpha'\alpha)a = \alpha'_{Ga} \circ F'\alpha a$ where $\alpha' : F' \to G'$, and $\alpha : F \to G$. Then $\forall a \in A$,

$$
\begin{aligned}
(\bar{1}_B\alpha)a &= (1_{1_B}\alpha)a \\
&= 1_{1_B}Ga \circ 1_B\alpha a \\
&= 1_{1_B Ga} \circ \alpha a \\
&= 1_{Ga} \circ \alpha a \\
&= \alpha a
\end{aligned} \tag{3.25}
$$

Hence $1_{1_B}\alpha = \alpha$. Now $\forall a \in A$,

$$
\begin{aligned}
(\alpha\bar{1}_A)a &= (\alpha 1_{1_A})a \\
&= \alpha_{1_A a} \circ F1_{1_A}a \\
&= \alpha a \circ F1_{1_A}a \\
&= \alpha a \circ F1_a \\
&= \alpha a \circ 1_{Fa} \\
&= \alpha a
\end{aligned} \tag{3.26}
$$

Which implies $\alpha\bar{1}_A = \alpha$, and the identity axioms are satisfied.

3.5 Equivalence

...though we cannot know these objects as things in themselves...

-Kant, Critique of Pure Reason

We can ask whether any two objects a and b in some arbitrary category are equal, but it turns a more important question is whether they are equivalent.

This means that $\exists f : a \to b, g : b \to a$ such that $gf = 1_a, fg = 1_b$, (this is an equivalence relation on all objects in the classical mathematical sense).

This is why: one of the 'motifs' of category theory is de-emphasise the objects and elevate the arrows between them, we consider the relations between objects to tell us all that is significant in the category. We cannot test for the equality of objects only with arrows, but we can test for equivalence.

In a 2-category, we can ask whether two arrows are equivalent way using the obvious construct. That is f and g are equivalent iff $\exists$ 2-cells $\alpha : f \to g$ and $\beta : g \to f$ such that $\beta\alpha = 1_f$ and $\alpha\beta = 1_g$. In general, if two arrows are equivalent, there is no reason to suppose they are equivalent in a unique way.

Given this construct, we can apply this to 'weakening' structures in the 2-category. For example, we can weaken the associativity law by not requiring that any two given bracketings of composable morphisms are equal, but asking for an equivalence, and in fact what is usually demanded is that the equivalence is unique. MacLane [Mac] shows that by assuming the pentagon identity, then there can only be ever one equivalence. (Actually [Mac] shows this only in the context of a monoidal category, but the proof generalises to any 2-category).

A systematic exploration of these ideas leads to notion of weak 2-category, first introduced by Benabou as a bicategory. The generalisation of these ideas leads to weak n-categories. The correct definition(s) of a weak n-category is the subject of current research in higher category theory [Len].

3.6 Monoids & Comonoids

The term 'Categorification' is an imprecise term coined by Crane [Crn] for the imprecise science of lifting mathematical tools in the 'universe' of sets into a category-theoretic world. Some general observations can be made. In set-theory, the notion of a 'set' is basic, and the functions between sets assume a subordinate role. In Category theory, this is subverted and the the notion of function becomes primary.

Monoids are basic to algebra in being one of the simplest algebraic structures. We categorify the notion (and find that its also basic in category theory). A monoid is a set A, closed under some associative binary operation $\mu : A \to A$, which we call multiplication and has a unit $e \in A$ for that operation.

Associativity means

$$\forall a, b, c \in A \quad (ab)c = a(bc) \tag{3.27}$$

And the unit axiom is

$$\forall a \in A \quad ae = a = ea \tag{3.28}$$

We can immediately express associativity equationally, that is

$$\forall a, b, c \in A \quad \mu(\mu(a, b), c) = \mu(a, \mu(b, c)) \tag{3.29}$$

Now for any set X, we have the function $id_X : X \to X$, which is just the identity on the elements. As we have only one set here, we shall just designate it by id. We re-express the equation as

$$\forall(a,b,c) \in A \times A \times A \quad \mu \circ (\mu \times id)(a,b,c) = \mu \circ (id \times \mu)(a,b,c) \tag{3.30}$$

This can now be interpreted as a commutative diagram

$$\begin{CD} A \times A \times A @>{\mu\circ(id\times\mu)}>> A \times A \\ @V{\mu\circ(\mu\times id)}VV @VV{\mu}V \\ A \times A @>>{\mu}> A \end{CD} \tag{3.31}$$

We've been able to categorify the notion of an associative multiplication without using the unit axiom, which is as it should be. The notion of a unit being unnecessary for the existance of such a multiplication. We now introduce the unit. First we choose some one-element set *, and designate it by 1. Then for any set X there are the unique left & right bijections:

$$1 \times X \xrightarrow{\lambda_X} X \xleftarrow{\rho_X} X \times 1 \tag{3.32}$$

given by $\lambda(*, a) = a$ and $\rho(a, *) = a$ As we have only the one set A, we shall just designate them λ and ρ. Now consider the following function

$$\eta : 1 \to A \tag{3.33}$$

it has only one value $\eta(*) \in A$, it 'picks' out a value in the set A. The unit axiom can be decomposed into two, the left and right unit axioms: $ea = a$ and $ae = a$. We can now categorify them.

$$
\begin{aligned}
ea &= \mu(e, a) \\
&= \mu \circ (\eta \times id)(*, a) \\
&= a \\
&= \lambda(*, a)
\end{aligned} \tag{3.34}
$$

and similarly for the right unit axiom. So we have:

$$
\begin{aligned}
\mu(\eta \times id) &= \lambda \quad \forall a \in A \\
\mu(id \times \eta) &= \rho \quad \forall a \in A
\end{aligned} \tag{3.35}
$$

We can express them as commutative diagrams

(3.36)

3.7 Monoidal Categories

To completely characterise the monoids categorically we need a 'categorification' of the cartesian product, and to do this we destill some properties. We've already introduced two, the bijections λ and ρ.

$$1 \times X \xrightarrow{\lambda_X} X \xleftarrow{\rho_X} X \times 1 \tag{3.37}$$

There is a subtlety in the cartesian product that was glossed over in the preceding discussion on monoids, that is the cartesian product is not strictly associative. However there is a standard isomorphism

$$\alpha_{X,Y,Z} : X(YZ) \to (XY)Z \tag{3.38}$$

For most purposes, its a subtlety that is usefully ignored, but it is fundamental to our discussion. It is easy to see that the following three diagrams commute

$$\begin{array}{ccc} 1 \times X & \xrightarrow{1\times F} & 1 \times X' \\ \downarrow{\scriptstyle \lambda_X} & & \downarrow{\scriptstyle \lambda'_X} \\ X & \xrightarrow[F]{} & X' \end{array} \qquad \begin{array}{ccc} X \times 1 & \xrightarrow{F\times 1} & 1 \times X' \\ \downarrow{\scriptstyle \rho_X} & & \downarrow{\scriptstyle \rho'_X} \\ X & \xrightarrow[F]{} & X' \end{array} \tag{3.39}$$

$$\begin{array}{ccc} X(YZ) & \xrightarrow{F(GH)} & X'(Y'Z') \\ \downarrow{\scriptstyle \alpha_{X,Y,Z}} & & \downarrow{\scriptstyle \alpha_{X',Y',Z'}} \\ (XY)Z & \xrightarrow[(FG)H]{} & (X'Y')Z' \end{array} \tag{3.40}$$

Where $X, Y, Z \& X', Y', Z'$ are sets, and $F : X \to X', G : Y \to Y', H : Z \to Z'$ are set maps. These diagrams implies that $\lambda, \rho \& \alpha$ are natural equivalences in the category Set. We illustrate why for α.

Define two functors $R, L : Set^3 \to Set$ by

$$
\begin{aligned}
R(X,Y,Z) &:= X(YZ) \quad R(F,G,H) := F(GH) \\
L(X,Y,Z) &:= (XY)ZL(F,G,H) := (FG)H
\end{aligned}
\tag{3.41}
$$

We show only R is a functor (L is similar).

$$
\begin{aligned}
R(F' \circ F, G' \circ G, H' \circ H) &= (F' \circ F)((G' \circ G)(H' \circ H)) \\
&= (F' \circ F)((G'H') \circ (GH)) \\
&= (F'(G'H')) \circ (F(GH)) \\
&= R(F', G', H') \circ R(F, G, H)
\end{aligned}
\tag{3.42}
$$

and the identity is satisfied since $R(1_X, 1_Y, 1_Z) = 1_X(1_Y 1_Z) = 1_{R_{X,Y,Z}}$ then the following diagrams are equal & commute, which shows that α is a natural transformation.

$$
\begin{array}{ccc}
R(X,Y,Z) & \xrightarrow{R(F,G,H)} & R(X',Y',Z') \\
\downarrow{\scriptstyle \alpha_{X,Y,Z}} & & \downarrow{\scriptstyle \alpha_{X',Y',Z'}} \\
L(X,Y,Z) & \xrightarrow{L(F,G,H)} & L(X',Y',Z')
\end{array}
\qquad
\begin{array}{ccc}
X(YZ) & \xrightarrow{F(GH)} & X'(Y'Z') \\
\downarrow{\scriptstyle \alpha_{X,Y,Z}} & & \downarrow{\scriptstyle \alpha_{X',Y',Z'}} \\
(XY)Z & \xrightarrow{(FG)H} & (X'Y')Z'
\end{array}
\tag{3.43}
$$

And it is a natural equivalence as α is invertible.

We also note, not only is there a standard mapping α, but between any two bracketings of an expression made up cartesian products and both left & right identities, there is a unique isomorphism. We wish to preserve this property.

The cartesian product is in particular, a map $Set^2 \to Set$, now both Set^2 and Set are categories, and the natural notion of a map between categories is a functor. Since the domain is a product category, we're actually looking for a bifunctor. (We have not discussed products in categories, see [Mac] for a clear discussion).

Summarising, we are looking for categories with a fixed bifunctor, which we call mutiplication, and that has left and right identities upto natural equivalence, and is also associative upto natural equivalence and furthermore any diagram put together with these natural transformations only should commute. We call this a monoidal category. Maclane proves in [Mac] that we need only show the pentagonal, and the triangle identities to obtain the latter property.

Before defining monoidal categories formally, we note that we can multiply functors into a category carrying an bifunctor. Suppose A carries the bifunctor $\otimes : A^2 \to A$, and we have functors $F, G : X \to A$ then

defining $(G \otimes F)x := Gx \otimes Fx$ on objects, and $(G \otimes F)f := Gf \otimes Ff$ on morphisms, we have

$$\begin{aligned} (G \otimes F)(gf) &= Ggf \otimes Fgf \\ &= GgGf \otimes FgFf \\ &= (Gg \otimes Fg)(Gf \otimes Ff) \\ &= (G \otimes F)g \circ (G \otimes F)f \end{aligned} \tag{3.44}$$

and

$$\begin{aligned} (G \otimes F)1_x &= G1_x \otimes F1_x \\ &= 1_{Gx} \otimes 1_{Fx} \\ &= 1_{Gx \otimes Fx} \\ &= 1_{(G \otimes F)x} \end{aligned} \tag{3.45}$$

Definition:

A monoidal category is

1. a category A

2. a bifunctor $\otimes : A \times A \to A$

3. an object $e \in A$

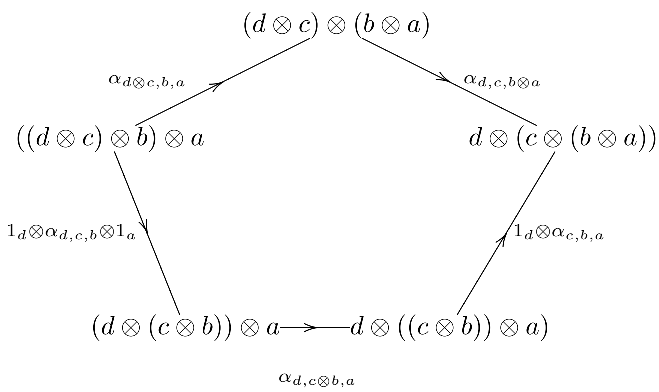
4. three natural isomorphisms

   $\alpha : (id \otimes id) \otimes id \to id \otimes (id \otimes id)$

   $\lambda : e \otimes id \to id$

   $\rho : id \otimes e \to id$

5. the three isomorphisms are 'coherent', that is α satisfies the pentagonal identity, and λ & ρ satisfy the triangle identities below.



(3.46)

When the pentagonal & triangle identities are vacuous, we call it a *strict* monoidal category. We can always lift a monoidal category A into a *single object* weak 2-category A', simply by fixing some object, and defining its 1-morphisms as objects of A, its 2-morphisms as the arrows of A, and vertical composition being the usual arrow composition in A, and the horizontal composition as the monoidal product, conversely we can give a sub-category generated by a some single object in a 2-category a monoidal structure. In a sense, a monoidal category is a fragment of a larger 2-category.

Chapter 4

String diagrams

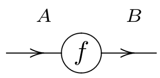
4.1 The 'Poincare Dual'

The usual dual of a commutative diagram simply reverses the direction of all the arrows in a diagram. However, there is another more 'geometric' way of taking a dual, and it reveals geometry, generally topological information in the the diagram itself.

This is the Poincare dual. Working in 1-space, this means points go to lines, and lines go to points. In a 1-category context this means the following diagram

$$A \xrightarrow{f} B \tag{4.1}$$

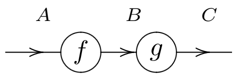
transforms into



and the composition of two arrows

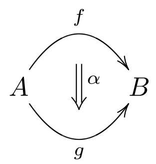
$$A \xrightarrow{f} B \xrightarrow{g} C \tag{4.2}$$

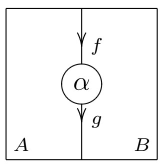
transforms into



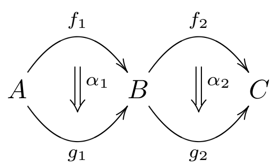
4.2 Strings in 2-Categories

Jumping up a dimension, that is working in 2-space, points will transform into planes, lines to lines, and planes to points. Hence a diagram in a 2-category like this

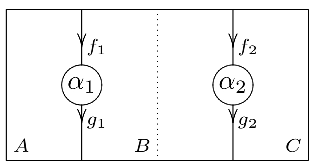




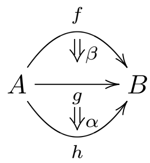
and horizontal composition



translates to



and vertical composition



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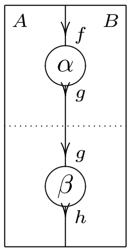


4.3 Unit & Counit

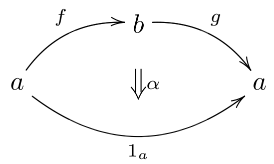
Let A be some arbitrary 2-category. Then a unit and counit are 2-morphisms formed by the following data:

Given two objects a, b and two morphisms $f : a \to b$, $g : b \to a$, a unit α and counit β are 2-morphisms satisfying:

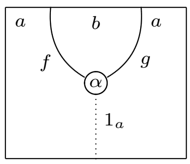
$$\begin{aligned} \alpha &: gf \to 1_a \\ \beta &: 1_b \to fg \end{aligned} \tag{4.3}$$

We translate this data into string diagrams.

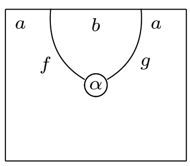
First, we express the unit as a globular diagram



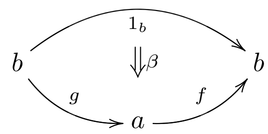
We take, the Poincare dual to obtain the string diagram.



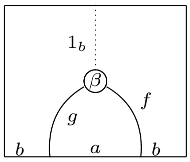
and remove the identity string.



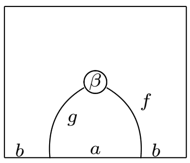
similarly for the counit β, the globular diagram is



which becomes

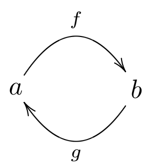


and without the identity morphism



4.4 Adjunctions & String Diagrams

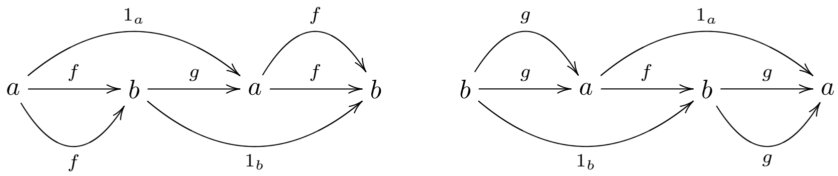
An adjunction in an arbitrary 2-category A, between two 1-morphisms $f : a \to b, g : b \to a$ requires a unit $\alpha : gf \to 1_a$, and a counit $\beta : 1_b \to fg$ to satisfy the following



satisfying

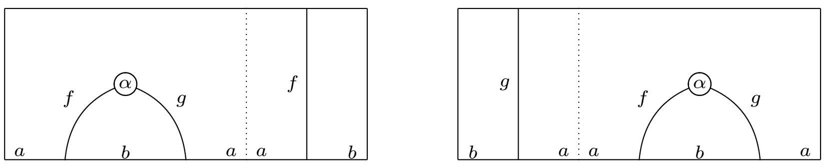
$$\begin{aligned} (1_f * \alpha)(\beta * 1_f) = 1_f \\ (\alpha * 1_g)(1_g * \beta) = 1_g \end{aligned} \tag{4.4}$$

where * denotes horizontal composition, and juxtaposition is vertical composition. As globular diagrams, it translates into



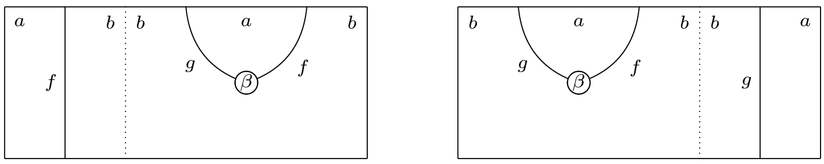
(4.5)

the top half of each diagram translates into string diagrams as



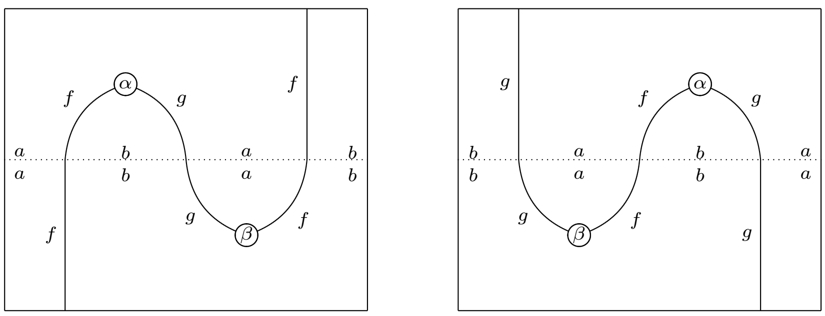
(4.6)

the bottom half of each diagram translates into string diagrams as



(4.7)

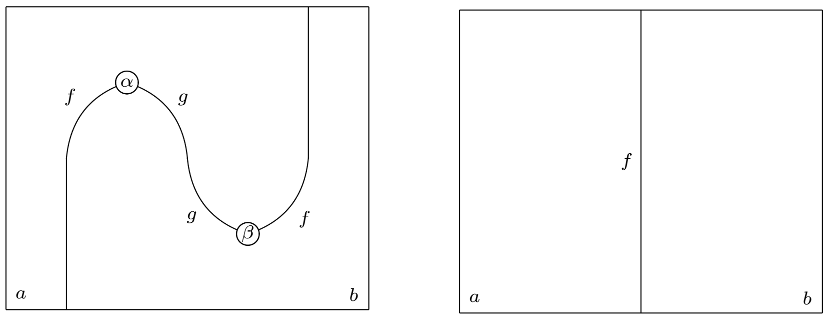
composing them



(4.8)

And the adjunction equation amount to a topological property in the string diagram - the string can be straightened, and we can imagine the unit and counit being pushed together and 'cancelling'.

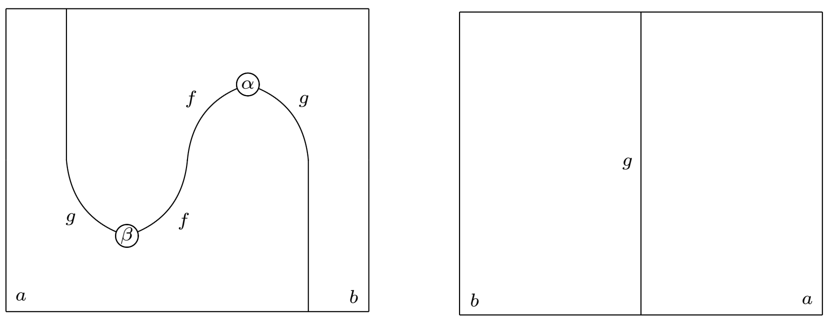
$$(1_f * \alpha)(\beta * 1_f) = 1_f \tag{4.9}$$



(4.10)

and similarly for the other equation

$$(\alpha * 1_g)(1_g * \beta) = 1_g \tag{4.11}$$



(4.12)

Chapter 5

Differential Geometry of Bundles

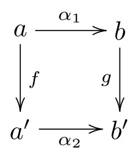
5.1 Arrow Category

Given a category A, we construct a new category $A^{[2]}$, called the category of arrows of A. Its objects are the morphisms of A, and its arrows are commuting squares of A.

More precisely, $ObjA^{[2]} = MorA$. Diagrammatically, we write for $f \in ObjA^{[2]}$, which is a morphism from say, $a \to a'$, where $a, a' \in ObjA$.

$$\begin{array}{c} a \\ \downarrow{\scriptstyle f} \\ a' \end{array} \tag{5.1}$$

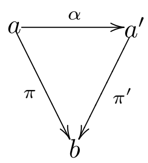
And given two objects $f : a \to a', g : b \to b'$ in $A^{[2]}$, a morphism $\alpha : f \to g$ in $MorA^{[2]}$ is a pair $(\alpha_1, \alpha_2) \in MorA^2$ such that the following square commutes.



(5.2)

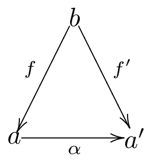
This definition is easily seen to satisfy the axioms of a category.

Fixing an object $b \in A$, there are two obvious subcategories, the over category $A^{[2]}/b$, where we choose $\alpha_2 = 1_b$, so that we have the following diagram for morphisms



(5.3)

and similarly for the under category $a/A^{[2]}$, by now choosing $\alpha_2 = 1_b$.



(5.4)

When we do this construction in Man, the category of manifolds, the arrow category is called the category of bundles $Bun := Man^{[2]}$, and choosing a specific manifold as the 'base' manifold, we get the category of bundles over M, $BunM := Bun/M$.

For any bundle $\pi : E \to M$, and any subset U of M, the manifold $E_U := \pi^{-1}E$ is also a bundle, and called the restriction of the bundle to U. When U is just a point p, then E_p is called the fibre of the bundle over p. The fibres organise themselves into a category called $FibE$, taking objects to be the fibres, and morphisms all fibre-preserving bundle maps. We note that any fibre-preserving bundle map, the map between the total spaces determines the map between base spaces uniquely. A map between bundles over M is always fibre-preserving, and in fact this property characterises these maps.

Given any manifold F, we can construct an obvious bundle over M, by choosing $E = FM$, and setting π to be the projection of FM onto its second factor. This called a trivial bundle. A bundle $E \in BunM$ is locally trivial, when there is some trivial bundle $E' = FM \in BunM$, such that $\forall p \in M$, there exists some open set U of M, such that $\phi_U : E_U \backsimeq E'_U$. F is called the standard fibre of the bundle, and ϕ a local trivialisation. (A cover U_i of M, where the bundle restriction E_i to U_i is trivial, is called a trivialising cover).

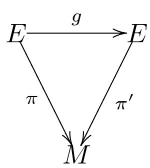
From now on, we only consider locally trivial bundles. We notice that that for any point $p \in M$, E_p must be isomorphic to the standard fibre F, and this in general is not true, unless we restrict to surjective submersions for bundle arrows, we will do so, and we will still denote this bundle as Bun.

Given a trivialising cover, we have a collection of trivial bundles and also a collection of transition functions $\phi_{UV} : (U \cap V)F \to F$ for trivialisations ϕ_U, ϕ_V, characterised by $\phi_U \circ \phi_V^{-1}(x,p) = (x, \phi_{UV}(x,p)) :=$, and since $\phi_{UV}(x,-)$ is a diffeomorphism, we may can cosider it to be a map $\phi_{UV} : (U \cap V) \to DiffF$ It is easily seen they satisfy the three conditions

$$\begin{aligned} &\phi_{UU} = 1 \\ &\phi_{VU} = \phi_{UV}^{-1} \\ &\phi_{UV} = \phi_{VW}\phi_{WU} \;\; on\ U \cap V \cap W \text{the cocycle condition} \end{aligned} \tag{5.5}$$

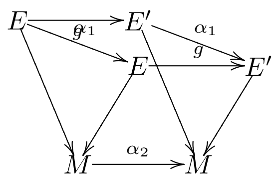
The converse is also true: Suppose we have a collection of trivial bundles whose transition functions satisfy the above, we can build a bundle, by essentially taking the disjoint union $\bar{E}$ of all $\alpha \times U_\alpha \times F$ with the relation: $(\alpha, x, v) \backsim (\beta, y, w)$ if and only if x=y and $\phi_{\beta\alpha}(x)v = w$. By the cocycle condition this is an equivalence relation, set $E := \bar{E}/\backsim$, and the projection $\pi : E \to M$, $p[(\alpha, x, v)] = x$. This is a bundle.

Now suppose the total space E is equipped with the action of some lie group G, then if it is fibre-preserving then any element $g \in G$, generates an automorphism of the bundle.



(5.6)

These too, organise themselves into a category whose morphisms are $(\alpha_1, \alpha_2, \theta) : (E, M, G) \to (E', M', G')$, where θ is a group homomorphism, and the following diagram commutes.



(5.7)

We specialise twice, first by fixing the group G, this $G - Bun$, the category of G-Bundles, and then by choosing the standard fibre to be G, and the action of G on the standard fibre to be just left translation, this is $G - PBun$ the category of principal G-bundles.

For $G - Pbun$, we have an alternative characterisation which is of importance in generalising our constructions. So we describe this in detail. $E|_U \to UG$ is a fibre-respecting diffeomorphism, hence we have a map $\tau : P \times_M P \to G$

satisfying $r(u_x, \tau(u_x, u'_x)) = u'_x$, when an implicit function theorem is available (this may not be true in certain infinite dimensional spaces).

The frame bundle of a n-dimensional manifold is naturally a $GL(n)$ principal bundle.

A bundle whose standard fibre is some vector space, is called a vector bundle. The standard example is any tangent bundle TM for any manifold M. The obvious constructions done fibre-wise on vector bundles work, and can be shown to result in well-defined bundles. Primarily $E \oplus E', E \otimes E', Hom(E, E')$ & $\wedge E, E*$. ([Mic] 6.7 demonstrates this using smooth functors and the characterisation of the vector bundle by cocycles).

The form bundle for a manifold M, is $\wedge T^*M$, its sections $\Omega M := \Gamma(\wedge T^*M)$ are the differential forms. Given some vector bundle E, we have the E-form bundle $\wedge T^*M \otimes E$, its sections $\Omega M := \Gamma(\wedge T^*M \otimes E)$ are called the differential forms valued in E.

We consider the endomorphism bundle $EndE := Hom(E, E)$. Any $T \in EndE$ defines a map $T : \Gamma R \to \Gamma E$, by defining the action fibrewise. In fact, any map $\Gamma R \to \Gamma E$ arises from such a T. That is $\Gamma(EndE) = End(\Gamma E)$ (where we think of ΓE as a $C^\infty M - Modules$).

Now suppose E is both a vector & G-bundle with standard fibre V a vector space, then any $g \in G$ defines an automorphism of the bundle, and in particular a map $E_p \to E_p$, that is we have a representation ρ of G on each fibre E_p, and using the standard isomorphism of the fibre with V, also an automorphism of V, also a representation ρ_U of G on V.

We say a linear transformation $T : E_p \to E_p$ is in G if it is of the form

$$[p,v]_U \rightarrow [p,gv]_U \tag{5.8}$$

for some g, where $[p,v]_U := \phi_U(x)$. This definition is independent of the choice of U, since suppose $p \in V$ also, then

$$[p,v]_U = [p,g_{VU}v]_V \tag{5.9}$$

and

$$[p,gv]_U = [p,g_{VU}gv]_V \tag{5.10}$$

so T is also given on $U \cap V$ by

$$[p,(g_{VU}v)]_V \rightarrow [p,(g_{VU}gg_{VU}^{-1})(g_{VU}v)]_V \tag{5.11}$$

The group of all such T is called the gauge group of the G-Bundle E.

5.2 Connections

To define a derivative of a vector bundle E, we need some way of taking the difference of arbitrary tangent vectors in TE. As we move from point p to point q in the bundle, a frame at p 'twists' into one at q; so the natural place to look at this is in the frame bundle FE. As principal bundles are the abstract generalisation of frame bundles, we begin here.

Let E' be a G principal fibre bundle. The tangent space along the fibre at the point $p \in E'$ is naturally isomorphic to TF, where F is the standard fibre.

These can be put together into a bundle called the vertical bundle VE. So the tangent space T_pE can be decomposed into $V_pE \oplus H_pE$ where H_pE is some complement of V_pE in T_pE. H_pE can be put together into a bundle called the horizontal bundle HE.

Now, the vectors in V_pE point along the fibre, so they 'connect' the tangent spaces in the the 'vertical' direction. We can use vectors in H_pE to connect tangent spaces in the horizontal direction. We can also describe the decomposition by a projection $\Phi : TE \to VE \subset TE$, such $Ker\Phi = HE$. We need to take into account the right action, the natural thing to do is to ask for G equivariance, in the following sense.

$$\begin{array}{ccc} T_uE & \xrightarrow{\Phi} & V_uE \\ {\scriptstyle T_u r^g}\downarrow & & \downarrow{\scriptstyle T_v r^g} \\ T_vE & \xrightarrow[\Phi]{} & V_vE \end{array} \tag{5.12}$$

where $v = (r^g)u = ug$. This is the definition of a principal connection.
A connection defines a structure called parallel transport on the principal bundle. This is a smooth functor $\tilde{\Phi} : PM \to FibE$. We show the existance of this functor locally, that is for sufficiently small paths. For the proof for arbitrary paths, see [Mic] Theorem 11.6.

Let $\alpha : p \to q \in PM$, and suppose $\tilde{\Phi} \in \Gamma\alpha$, then $\Phi(\frac{\tilde{\Phi}}{dt}) = 0$ is a first order ODE, and for sufficently small α and for a point u in the fibre E_p over p, it is a standard result that a solution with initial condition $\tilde{\Phi} = u$ exists, and we have a well-defined element $v \in E_q$.

The functor is easily seen to be reparametrisation invariant. Not all connections come from such functors, and it is possible to characterise such functors.

Parallel transport functors without the smoothness condition are important in Loop Quantum Gravity.

We can now induce a general connection and parallel transport on an associated bundle, using its natural bundle $\bar{\pi} : E \times F \to E \times_G F$. Consider the following diagram:

$$\begin{array}{ccc} TE \times TF & \xrightarrow{\Phi \times Id} & TE \times TF \\ {\scriptstyle T\bar{\pi}}\downarrow & & \downarrow{\scriptstyle T\bar{\pi}} \\ TE \times_T GTF & \xrightarrow[\bar{\Phi} \times Id]{} & TE \times_T GTF \end{array} \tag{5.13}$$

[Mic] 11.8, proves that $\bar{\Phi}$ is well-defined, that its image is $VE' = E \times_G TF$, so that it does define a general connection on the associated bundle. We also have $\bar{\pi}(HE \times TF) \subset HE' = ker\bar{\Phi}$. By a similar construction to that for a principal bundle, we can now construct a parallel transport functor $P\bar{h}i' : PM \to FibE'$ for this connection. By uniqueness of parallel transports, we have the following commutative diagrams relating the two connections:

$$\begin{array}{ccc} E \times FE' & \xrightarrow{\bar{\pi}} & E' \\ {\scriptstyle \Phi\alpha \times id_F}\downarrow & & \downarrow{\scriptstyle \tilde{\Phi}'\alpha} \\ E \times F & \xrightarrow[\bar{\pi}]{} & E' \end{array} \tag{5.14}$$

When the associated bundle is a vector bundle, then the induced connection is fibre-wise linear: $\Phi'_p : T_pE' \to T_pE'$ is a linear map. In fact, [Mic] 11.11 shows that a linear connection is always induced from a unique connection on the principal bundle.

We can now define the covariant derivative, $\nabla : \Gamma M \to End\Gamma E = \Gamma End\ E$.

Choose $X \in T_pM$, then there exists a path $\alpha \in PM$ such that $\alpha(0) = p$ and $\dot{\alpha}(0) = X$. Now choose $\phi \in \Gamma\alpha$, and define

$$\nabla_X \phi := \frac{d\phi_X}{dh}|_{h=0} \tag{5.15}$$

where we have defined $\phi_X(h) := \Phi'(\alpha_{[0,h)})\phi[\alpha(h)]$. The usual properties of the covariant derivitive follow ([Kob] III proposition 1.1):

1. $\nabla_{X+Y}\phi = \nabla_X\phi + \nabla_Y\phi$

2. $\nabla_{\lambda X}\phi = \lambda\nabla_X\phi$ (λ is a constant)

3. $\nabla_X(\phi + \psi) = \nabla_X\phi + \nabla_X\psi$

4. $\nabla_X(\lambda\phi) = \lambda\nabla_X\phi + (X\lambda)\phi$ (λ is a functional)

where $X, Y \in T_pM$, $\phi, \psi \in \Gamma_U E$ in some neighbourhood U of p. We show only the Liebniz property. We have $\Phi(\alpha)[\lambda(h)\phi(h)] = \lambda(h)\Phi(\alpha)$ by fibre-wise linearity, and taking the derivative as in the definition, property 4 follows.

We consider the space of all covariant derivative $\nabla : \Gamma M \to \Gamma End E$, this is $[\Gamma M, \Gamma End\ E] := Hom[\Gamma M, \Gamma End\ E]$, the space of all linear mappings between the two spaces, then

$$\begin{aligned} [\Gamma M, \Gamma End\ E] &\cong \Gamma[TM, [E, E]] \\ &\cong \Gamma[TM \otimes E, E] \\ &\cong \Gamma[E \otimes TM, E] \\ &\cong \Gamma[E, [TM, E]] \\ &\cong \Gamma[E, T^*M \otimes E] \\ &\cong [\Gamma E, \Gamma T^*M \otimes E] \end{aligned} \tag{5.16}$$

using the universal property of the tensor product, and for vector bundles E, E', that $[E^*, E'] \cong [E, E']$, and that $\Gamma[E, E'] \cong [\Gamma E, \Gamma E']$ which can be shown be taking taking local trivialisations and a partition of unity.

Now bundle valued forms are, by definition $\Omega^k E := \Gamma(\wedge^k T^*M \otimes E)$, hence $\Omega^0 E = \Gamma E$, and $\Omega^1 E = \Gamma(T^*M \otimes E)$. So we can rephrase the covariant derivative as a mapping

$$\Omega^0 E \xrightarrow{\nabla} \Omega^1 E \tag{5.17}$$

This looks very much like the first map in the exterior derivative complex for form bundles:

$$\Omega^0 M \xrightarrow{d} \Omega^1 M \Omega^2 M \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n M \tag{5.18}$$

In the next section, we show that can indeed form such an an extension, this is the exterior covariant derivative for a given connection.

5.3 Connection and Curvature forms

The vertical space of a G-principal bundle is isomorphic to the lie algebra $Lie; G$ of G, we use the fundamental vector field mapping ([Mic] 5.13) to produce this mapping. Given a right action $r : MG \to M$, we define $\eta : Lie; G \to \Gamma M$ by

$$(\eta X)(p) := (T_e r_p)X \tag{5.19}$$

where $p \in M$, and $r_p : G \to M$ by $r_p g := pg$. Then η is a linear G-equivariant mapping, and using the right action of a principal bundle we get actually get a linear G-equivariant isomorphism between the lie algebra of the structure group and the vertical space.

As both the connection Φ, and curvature $R(X,Y) := \Phi(\bar{\Phi}X, \bar{\Phi}Y)$ are vertically valued, we get a connection 1-form ω, and curvature 2-form Ω, both valued in $LieG$ by

$$\begin{aligned} \omega &= \eta^{-1}\Phi \\ \Omega &= \eta^{-1}R \end{aligned} \tag{5.20}$$

We also define $\bar{\Phi}^* : \Omega(E,V) \to \Omega(E,V)$ for some trivial vector bundle V, by

$$(\bar{\Phi}^*\phi)(X_1, \cdots, X_n) := \phi(\bar{\Phi}X_1, \cdots, \bar{\Phi}X_2) \tag{5.21}$$

By definition of the wedge product for forms, and the fact that $\bar{\Phi}$ is a projection, it is easy to establish the following properties:

$$\begin{aligned} \bar{\Phi}^* \circ \bar{\Phi}^* &= \bar{\Phi}^* \\ \bar{\Phi}^*(\phi \wedge \psi) &= \bar{\Phi}^*\phi \wedge \bar{\Phi}^*\psi \\ i_X(\bar{\Phi}^*\phi) &= 0 \ \ for\ X \ \in \ VE \end{aligned} \tag{5.22}$$

hence $\bar{\Phi}^*$ is actually a projection of $\Omega(E,V)$ onto the space of horizontal forms $\Omega_{hor}(E,V)$.

We define the exterior covariant derivative $d_\omega : \Omega^k(E,U) \to \Omega^{k+1}(E,U)$ for a connection by (labelling the derivative with the connection form ω, rather than the connection Φ itself):

$$d_\omega\phi = \bar{\Phi}^*(d\phi) \tag{5.23}$$

We quote a lemma in [Mic] 11.1 identifying some properties of the connection form:

1. the connection form reproduces the generators of the fundamental vector fields: $\omega[\eta_X(u)] = X$

2. for the Lie derivative, we have: $L_{\eta X}\omega = -ad(X)\omega$

and derive the following:

1. ω kills horizontal fields, and Ω kills vertical fields

2. Maurer-Cartan: $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ (the bracket is for $LieG$ valued forms)

3. $d_\omega(\phi \wedge \psi) = d_\omega\phi \wedge \bar{\Phi}^*\psi + (-1)^{|\phi|}\bar{\Phi}^*\phi \wedge d_\omega\psi$

4. $d_\omega\omega = \Omega$

5. $d_\omega\Omega = d_\omega^2\omega = 0$ the Bianchi identity

1. follows as Φ and $\bar{\Phi}$ are projections onto the vertical and horizontal spaces respectively.

2. We show that the Maurer-Cartan formula holds for both vertical and horizontal vector fields. First the curvature $R(X, Y) := \Phi(\bar{\Phi}X, \bar{\Phi}Y)$ vanishes on vertical fields, since $\bar{\Phi}$ is a projection onto the horizontal field, and we have for a vertical field X

$$\begin{aligned} i_{\eta X} &= i_{\eta X}d\omega + \frac{1}{2}[i_{\eta X}\omega, \omega] - \frac{1}{2}[\omega, i_{\eta X}\omega] \\ &= L_{\eta X}\omega - d(i_X\omega) + \frac{1}{2}[X, \omega] - \frac{1}{2}[X, \omega] \\ &= -(adX)\omega + (adX)\omega \end{aligned} \tag{5.24}$$

using the derivation property of the insertion operator i_Z, that in general $L_Z\alpha = i_Z\alpha + di_Z\alpha$, and the definition of the lie bracket for $LieG$ valued forms.

and for horizontal fields X, Y, we have

$$\begin{aligned} R[X,Y] &= \Phi[\bar{\Phi}X, \bar{\Phi}Y] \\ &= \Phi[X,Y] \\ &= \eta_{\omega([X,Y])} \end{aligned} \tag{5.25}$$

as $\bar{\Phi}$ is a projection onto the horizontal space, and

$$\begin{aligned} d(\omega + \frac{1}{2}[\omega,\omega](X,Y) &= X\omega Y - Y\omega X - \omega([X,Y]) + [\omega X, \omega Y] \\ &= -\omega([X,Y]) \end{aligned} \tag{5.26}$$

since ω kills horizontal fields.

3. is obvious, given $\bar{\Phi}^*$ distributes over the wedge product.

4. We have

$$\begin{aligned} d_\omega \omega(X,Y) &= (\bar{\Phi}^* d\omega)(X,Y) \\ &= d\omega(\bar{\Phi}X, \bar{\Phi}Y) \\ &= \bar{\Phi}X\omega\bar{\Phi}Y - \bar{\Phi}Y\omega\bar{\Phi}X - \omega([\bar{\Phi}X, \bar{\Phi}Y]) \\ &= -\omega([\bar{\Phi}X, \bar{\Phi}Y]) \\ &= -\eta^{-1}\Phi([\bar{\Phi}X, \bar{\Phi}Y]) \\ &= \Omega(X,Y) \end{aligned} \tag{5.27}$$

5. and using the Maurer-Cartan formula, we establish

$$\begin{aligned} d_\omega \Omega &= d_\omega (d_\omega + \frac{1}{2}[\omega, \omega]) \\ &= \bar{\Phi}^* dd\omega + \frac{1}{2} \bar{Phi}^* d[\omega, \omega] \\ &= \frac{1}{2} \bar{\Phi}^* ([d\omega, \omega] - [\omega, d\omega]) \\ &= \bar{\Phi}^* ([d\omega, \omega]) \\ &= [\bar{\Phi}^* d\omega, \bar{\Phi}^* \omega]) \\ &= 0 \;\; since \; \bar{\Phi}^* \omega = 0 \end{aligned} \tag{5.28}$$

5.4 Vector Bundles

We refer to [Mic] 11.11-15 for proofs of the following assertions.

First, given a principal bundle E over some manifold M, and an action ρ of the structure group G on some finite-dimensional vector space U, we can construct an associated vector bundle $E' := E \times_\rho U$ over the same base manifold.

There is a canonical isomorphism $q^U : \Omega(M, E') \to \Omega_{hor}(E, U)^G$ between forms valued in this vector bundle, and horizontal & G-equivariant forms on the principal bundle and valued in the vector space.

Recall that also there is a canonical isomorphism of connections Φ on the principal bundle E, and linear connections Φ' on the associated vector bundle E', and the linear connection induces a a covariant derivative operator ∇, this can be used to define the curvature $R^{E'} \in \Omega^2(M, End\, E')$ of the linear connection, and also an exterior covariant derivative d_∇.

$$R^{E'}(X, Y)Z := ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z \tag{5.29}$$

$$d_\nabla \alpha(X_0, \cdots, X_p) = \sum_{i=0}^{p} (-1)^i \nabla_{X_i} \alpha(X_0, \cdots, \hat{X}_i, \cdots, X_p) + \sum_{0 \leqslant i < j \leqslant p} (-1)^{i+j} \alpha([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_p) \tag{5.30}$$

where X, Y, X_i are vector fields on M, and $Z \in \Gamma E'$.

The exterior covariant derivative d_∇ has the following properties:

$$\begin{aligned} (d_\nabla s)X &= \nabla_X s \\ d_\nabla(\alpha \wedge \beta) &= d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta \\ d_\nabla^2 \beta &= R^{E'} \wedge \beta \\ d_\nabla R^{E'} &= 0 \end{aligned} \tag{5.31}$$

where $s \in \Omega^0(M, E') = \Gamma E', X \in \Gamma M, \alpha \in \Omega M$ and $beta \in \Omega(M, E')$. The first two properties are generalisations of the usual covariant derivative operator, the fourth is the Bianch identity. Note that the wedge in third property is between E' and $End\ E'$ valued forms.

They are related to the connection form $\Omega \in \Omega(E, Lie\ G)$ and exterior covariant derivative d_ω on the principal bundle as follows:

$$\begin{array}{ccc} \Omega(M, E') & \xrightarrow{q^U} & \Omega_{hor}(E, U)^G \\ \downarrow d_\nabla & & \downarrow d_\nabla \\ \Omega(M, E') & \xrightarrow[q^U]{} & \Omega_{hor}(E, U)^G \end{array} \tag{5.32}$$

$$q^{End\ E'} R^{E'} = \rho' \circ \Omega \tag{5.33}$$

where $\rho' := T_e\rho : Lie\ G \to End(U, U)$ is the derivative of the representation at the identity.

5.5 Chern-Simons Theory

The Yang-Mills equation is

$$S_{YM} := \int_M tr(F \wedge *F) \tag{5.34}$$

It is explicitly dependent on the metric through the Hodge dual. Now n-forms are naturally integrated on n-dimensional space, and the *Chern* form F^n has degree $2n$, and hence can be used to define a metric-free action on a $2n$-dimensional spacetime.

$$S_C(A) := \int_M tr(F^n) \tag{5.35}$$

It turns out that every vector potential is a critical point of the action, by the following argument using the Bianchi identity

$$\begin{aligned}
\delta S_C &= n \int_M tr(\delta F \wedge F^{n-1}) \\
&= n \int_M tr(d_D \delta A \wedge F^{n-1}) \\
&= n \int_M tr(\delta A \wedge d_D F^{n-1}) \\
&= n \int_M tr(\delta A \wedge (d_D F \wedge F^{n-2} + F \wedge d_d F \wedge F^{n-3} + \cdots)) \\
&= 0
\end{aligned} \tag{5.36}$$

This means that the action is actually an invariant of the bundle. In fact, the Chern form defines a cohomology class in H^{2n}: we show the form is closed, and that it changes by exact forms.

$$
\begin{aligned}
d\,tr(F^n) &= tr(d_D F^n) \\
&= tr(d_D F \wedge F^{n-1} + F \wedge d_D F \wedge F^{n-2} + \cdots) \\
&= 0
\end{aligned}
\tag{5.37}
$$

by the Bianchi identity, and so it is closed. Using the graded cyclic property of the trace to gather all the curvature terms, we have

$$
\begin{aligned}
\delta tr(F^n) &= tr(\delta F^n) \\
&= tr(\delta F \wedge F^{n-1} + F \wedge \delta F \wedge F^{n-2} + \cdots) \\
&= n\,tr(\delta F \wedge F^{n-1}) \\
&= n\,tr(d_D \delta A \wedge F^{n-1}) \\
&= n\,tr(d_D(\delta A \wedge F^{n-1})) \\
&= n\,d\,tr(\delta A \wedge F^{n-1})
\end{aligned}
\tag{5.38}
$$

We set $\delta A := A' - A$ and $A_s := A + s\delta A$,for A' any other vector potential, and let F' and F_s be the curvature of A' and A_s respectively, we calculate:

$$
\begin{aligned}
tr(F'^n) - tr(F^n) &= \int_0^1 \frac{d}{ds} tr(F_s^n) ds \\
&= n \int_0^1 d\,tr(\delta A \wedge F_s^n) ds \\
&= n\,d \int_0^1 tr(\delta A \wedge F_s^n) ds
\end{aligned}
\tag{5.39}
$$

and the difference $tr(F'^n) - tr(F^n)$ is exact.

When we work over a trivial vector bundle, the form itself is exact, and an explicit expression for the $(n-1)$-form can be written down whose exterior derivative is the Chern form F^n, it is named the Chern-Simons form.

In detail, the space of connections is an affine space modelled on vector potentials, that is after choosing a specific connection D_0, all others are $D_0 + A$ for some vector potential A. Since the given bundle is trivial, the standard flat connection is available, and we make this our choice. When $A = 0$, the curvature of $D_0 + A$ is zero, and so the Chern form $tr\ F^n$ is zero, and hence exact; but we've already shown that changing the vector potential changes the Chern form by exact forms, so it must remain exact.

For simplicity, we derive an expression for the form F', such that $dF' = tr(F^2)$, a similar argument works for all n. Let $A_s := sA$, and let $F_s := d\ A_s + A_s \wedge A_s = s(dA) + s^2(A \wedge A)$ be its curvature, then

$$\begin{aligned} tr(F \wedge F) &= \int_0^1 \frac{d}{ds} tr(F_s \wedge F_s) ds \\ &= 2 \int_0^1 tr(\frac{d}{ds} F_s \wedge F_s) ds \\ &= 2d \int_0^1 tr(A \wedge F_s) ds \\ &= 2d \int_0^1 tr(A \wedge (s(dA) + s^2(A \wedge A))) ds \\ &= 2d \int_0^1 tr(sA \wedge dA + s^2 A \wedge A \wedge A) ds \\ &= d\ tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \end{aligned} \tag{5.40}$$

We can think of the Chern-Simon form as a 'boundary' term: form a trivial vector bundle over any space, and by choosing any connection over this bundle, we can construct a Chern form, and integral of this over the space , by Stokes theorem, equals the integral of the Chern-Simons form over the boundary.

We can use the Chern-Simons form to write an action

$$S_{CS}(A) := \int_M tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{5.41}$$

The theory is less trivial than the one given by the Chern action, but still not very interesting - the critical points of the actions are all flat vector potentials, by the following calculation:

$$\begin{aligned} \delta S_{CS} &= \delta \int_M tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \\ &= \int_M tr(\delta A \wedge dA + A \wedge d\delta A + \frac{2}{3}(\delta A \wedge A \wedge A + A \wedge \delta A \wedge A + A \wedge A \wedge \delta A) \\ &= \int_M tr(\delta A \wedge dA + dA \wedge \delta A + 2(A \wedge A \wedge \delta A) \\ &= 2 \int_M tr((dA + A \wedge A) \wedge \delta A) \\ &= 2 \int_M tr(F \wedge \delta A) \end{aligned} \tag{5.42}$$

and this only vanishes for all variations δA, if the curvature of the connection $F := dA + A \wedge A$ is zero.

The Chern-Simons action is invariant under all (orientation preserving) diffeomorphisms due invariance of the integral by pullbacks, and it is also invariant under *small* gauge transformations - those that are connected to the identity by the following argument.

Consider a 1-parameter family of gauge transformations g_s, going through the identity at $s = 0$. Then the gauge-transformed vector potential is $A_s := g_s A g^{-1} + g_s d(g_s^{-1})$, we show that

$$\frac{d}{ds}S_{CS}(A_s) = 0 \tag{5.43}$$

We need only show this at $s = 0$, then

$$\begin{aligned}\frac{d}{ds}A_s|_{s=0} &= \frac{d}{ds}(g_s A g_s^{-1} + g_s d(g_s^{-1}))|_{s=0} \\ &= (TAg_s^{-1} - g_s AT + Td(g_s^{-1} - g_s dT)|_{s=0} \\ &= [T, A] - dT\end{aligned} \tag{5.44}$$

where we have defined $T := \frac{d}{ds}g_s|_{s=0}$, and it is easy to see by differentiating the identity, that $\frac{d}{ds}g_s^{-1}|_{s=0} = -T$. Then

$$\begin{aligned}\frac{d}{ds}A_s|_{s=0} &= \frac{d}{ds}(g_s A g_s^{-1} + g_s d(g_s^{-1}))|_{s=0} \\ &= (TAg_s^{-1} - g_s AT + Td(g_s^{-1} - g_s dT)|_{s=0} \\ &= [T, A] - dT\end{aligned} \tag{5.45}$$

And noting that $\int tr(A \wedge A \wedge [T, A]) = 0$ by expanding the bracket and using the graded cyclic property of the trace, we have finally using stokes theorem

$$\begin{aligned}\frac{d}{ds}S_{CS}(A_s)|_{s=0} &= 2\int tr([T, A] \wedge dA + A \wedge A \wedge ([T, A] - dT)) \\ &= 2\int tr([T, A] \wedge dA - A \wedge A \wedge dT) \\ &= 2\int tr(d(A \wedge T \wedge A)) \\ &= 2\int d(tr(A \wedge T \wedge A)) \\ &= 0\end{aligned} \tag{5.46}$$

Under *large* gauge transformations - that is for those that are not connected to the indentity, the Chern-Simons action, although not invariant changes only by an integer multiple of $8\pi^2$, this is a consequence of the integrality of the second Chern class ([Baz] II.4).

Chapter 6

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